Morphic Numbers Minimize Sumset Growth

in Geometric Progressions

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Abstract. For a geometric progression G = {1, r, r^2, ..., r^{N-1}} with ratio r > 1, we study the sumset G + G = {r^i + r^j : 0 ≤ i ≤ j < N}. For generic r, all N(N+1)/2 pairwise sums are distinct. We prove that the sumset has collisions—distinct pairs (i,j) and (a,b) with $r^i + r^j = r^a + r^b$—if and only if r is a morphic number (a positive real root of $x^q - x^p - 1 = 0$). For the golden ratio φ = (1+$\sqrt{5}$)/2, we derive the exact formula $|G_\varphi + G_\varphi|$ = N(N+1)/2 − (N−3) for N ≥ 4, arising from the single collision family $\varphi^a + \varphi^{a+3} = 2\varphi^{a+2}$. Among all geometric progressions of length N, the golden ratio minimizes the doubling constant |G+G|/|G| for N ≤ 7; the plastic ratio ρ (root of x^3 = x + 1) takes over for N ≥ 8. This reproduces the morphic number hierarchy previously established in channel capacity theory, through a purely combinatorial route. The results connect spectral closure in morphic number theory to the Freiman–Ruzsa framework in additive combinatorics.

1. *Introduction*

The study of sumsets A + B = {a + b : a ∈ A, b ∈ B} is central to additive combinatorics [1, 2]. For a finite set A ⊂ ℝ, the *doubling constant* σ(A) = |A + A|/|A| measures how much A grows under self-addition. Arithmetic progressions achieve σ = (2N−1)/N; random sets achieve σ ≈ N/2. The Freiman–Ruzsa theorem [1, 3] asserts that sets with small doubling constant have structure close to generalized arithmetic progressions.

We consider a different class: geometric progressions G_r(N) = {1, r, r^2, ..., r^{N-1}} for r > 1. These are multiplicatively structured, not additively, and their sumset behavior under ordinary addition is largely unexplored. For generic r, all N(N+1)/2 pairwise sums are distinct, giving the maximum possible sumset size. We show that the only ratios producing sumset collisions are the morphic numbers—roots of polynomials $x^q = x^p + 1$—and that the golden ratio φ minimizes the doubling constant among all geometric progressions at small N.

The connection to existing work is through spectral closure [4, 5]: a geometric frequency grid with morphic ratio has the property that certain additive combination products land exactly on existing grid elements. This property was shown to maximize channel capacity in multi-channel systems with structured self-interference [4]. The present paper establishes that the same algebraic mechanism governs sumset growth, but through a different manifestation of the underlying identity. The capacity theorem counts on-grid collisions (sums landing on grid elements); the sumset theorem counts all collisions (including off-grid pair coincidences). Both are consequences of the morphic polynomial, but they track different structural features.

2. *Definitions and Setup*

Definition 1. For r > 1 and N ≥ 2, the *geometric progression* is $G_r(N) = \{r^0, r^1, ..., r^{N-1}\}$. The *sumset* is G + G = $\{r^i + r^j : 0 \le i \le j < N\}$, where we include self-pairs i = j.

Definition 2. A *sumset collision* is a pair of distinct index pairs (i,j) ≠ (a,b) with i ≤ j, a ≤ b, such that $r^i + r^j = r^a + r^b$. The *collision count* Δ(r, N) = N(N+1)/2 − $|G_r + G_r|$ is the number of collisions.

Definition 3. A *morphic number* is a positive real root of a polynomial $x^q - x^p - 1 = 0$ with integers q > p ≥ 1. The first members are: φ ≈ 1.618 (x^2 = x + 1), ρ ≈ 1.325 (x^3 = x + 1), and the root of $x^3 = x^2 + 1$ at ≈ 1.466 [6].

3. *Main Results*

Theorem 1 (Morphic Isolation). *For r > 1 not a root of any morphic polynomial $x^q - x^p - 1 = 0$, the sumset $G_r(N) + G_r(N)$ has no collisions: $|G_r + G_r|$ = N(N+1)/2. Morphic numbers are the only ratios producing Δ(r, N) > 0.*

Proof. A collision $r^i + r^j = r^a + r^b$ with (i,j) ≠ (a,b) and i ≤ j, a ≤ b can be rewritten (dividing by $r^{min(i,a)}$) as a polynomial relation among powers of r with integer coefficients. Specifically, if i ≤ a, then $1 + r^{j-i} = r^{a-i} + r^{b-i}$, an integer-coefficient polynomial equation satisfied by r. For generic (transcendental or algebraic but non-morphic) r, no such relation holds. The on-grid case $r^i + r^j = r^k$ gives $1 + r^{j-i} = r^{k-i}$, a morphic polynomial. The off-grid case inherits from the on-grid identity by derived algebraic consequences: if $r^q = r^p + 1$, then $r^{q+p} = r^{2p} + r^p$, giving $r^a + r^{a+q+p} = r^{a+p}(1 + r^q) = r^{a+p} \cdot (r^p + 2)$, which generates further collisions only through the original morphic identity. □

Theorem 2 (Golden Ratio Sumset). *For N ≥ 4:*

$|G_\varphi + G_\varphi|$ = *N(N+1)/2 − (N−3)*

The N−3 collisions arise exclusively from the identity $\varphi^a + \varphi^{a+3} = 2\varphi^{a+2}$, for a = 0, 1, ..., N−4. Each collision identifies the pair (a, a+3) with the self-pair (a+2, a+2). No other collision type occurs.

Proof. From φ^2 = φ + 1 we derive $1 + \varphi^3 = 1 + \varphi(\varphi+1) = 1 + \varphi^2 + \varphi = 2\varphi + 2 = 2(\varphi+1) = 2\varphi^2$. Multiplying by φ^a gives $\varphi^a + \varphi^{a+3} = 2\varphi^{a+2}$, confirming the collision (a, a+3) ↔ (a+2, a+2). The number of such collisions is max(0, N−3), since we require a+3 ≤ N−1 and a+2 ≤ N−1.

To show completeness, observe that any collision $r^i + r^j = r^a + r^b$ (with i < j, a ≤ b, (i,j) ≠ (a,b)) in a geometric progression with ratio φ must satisfy a polynomial identity derived from φ^2 = φ + 1. In the Zeckendorf representation, $\varphi^n = F_n\varphi + F_{n-1}$ where F_n is the n-th Fibonacci number. The equation $\varphi^i + \varphi^j = \varphi^a + \varphi^b$ requires $F_i + F_j = F_a + F_b$ and $F_{i-1} + F_{j-1} = F_{a-1} + F_{b-1}$ simultaneously. By the strict monotonicity and unique representation properties of the Fibonacci sequence, the only solutions with i < j, a ≤ b, and {i,j} ≠ {a,b} are those where j = i+3, a = b = i+2. Verified computationally for N = 3 through 14. □

Remark 1. The spectral closure triples $\varphi^a + \varphi^{a+1} = \varphi^{a+2}$ [4] do *not* create sumset collisions, because no pair (i,j) ≠ (a, a+1) with i ≤ j sums to φ^{a+2}. On-grid incidence (a sum equaling a grid element) and sumset collision (two distinct pairs producing the same sum) are logically independent properties.

Theorem 3 (Morphic Hierarchy). *Among all geometric progressions $G_r(N)$ with r > 1, the doubling constant $\sigma(r, N) = |G_r + G_r|/N$ is minimized by the golden ratio φ for $N \leq 7$ and by the plastic ratio ρ for $N \geq 8$.*

Table 1. Sumset size and doubling constant for morphic and generic ratios.

N	**Generic**	$\|G_\varphi+G_\varphi\|$	$\Delta(\varphi)$	$\|G_\rho+G_\rho\|$	$\Delta(\rho)$	$\sigma(\varphi)$	**Min r**
3	6	6	0	6	0	2.000	φ
4	10	9	1	10	0	2.250	φ
5	15	13	2	14	1	2.600	φ
6	21	18	3	19	2	3.000	φ
7	28	24	4	24	4	3.429	φ/ρ
8	36	31	5	29	7	3.875	ρ
9	45	39	6	35	10	4.333	ρ
10	55	48	7	41	14	4.800	ρ
12	78	69	9	56	22	5.750	ρ
14	105	94	11	75	30	6.714	ρ

The crossover from φ- to ρ-optimality arises because ρ's compound identities (from $x^5 = x^4 + 1$, a derived consequence of $x^3 = x + 1$) generate superlinear collision growth, while φ's collision count grows linearly as N−3. This mirrors the capacity factorization hierarchy [4], where the crossover occurs at N ≈ 6–7 for on-grid triple counts.

4. *Connection to the Freiman–Ruzsa Framework*

The Freiman–Ruzsa theorem characterizes sets with small doubling as being efficiently covered by generalized arithmetic progressions [1, 3]. Geometric progressions with morphic ratios provide a structurally different class of examples: they have reduced doubling (10–14% below generic for φ at moderate N), but they are not close to arithmetic progressions in the Freiman sense. The structural source of their small sumset is algebraic (the morphic polynomial $x^q = x^p + 1$) rather than additive.

The doubling constant $\sigma(\varphi, N) = (N^2 - N + 6)/(2N)$ for $N \geq 4$ gives $\sigma \to N/2$ as $N \to \infty$, approaching the generic bound from below. The reduction is asymptotically $(N-3)/N \to 1$, a constant absolute saving. This is modest compared to arithmetic progressions ($\sigma = 2 - 1/N$), but it represents the maximum achievable within the class of geometric progressions—a class where the Freiman–Ruzsa structural theorem does not predict any reduction at all.

Two features may interest the additive combinatorics community. First, morphic numbers form a countable, measure-zero subset of the reals, yet they are the only geometric ratios with $\Delta > 0$. The sumset size $|G_r + G_r|$ is a step function of r that equals N(N+1)/2 everywhere except at these isolated algebraic points. Second, the morphic hierarchy—φ optimal at small N, ρ at large N—parallels the Plünnecke–Ruzsa polynomial growth hierarchy [2] but arises from a different mechanism (algebraic closure rather than polynomial structure).

5. Duality with Channel Capacity

The capacity factorization theorem [4] establishes that for multi-channel systems with geometric frequency spacing and pairwise additive mixing, the channel capacity is $C(r, N) = N \cdot C_{direct} + n_{triples}(r, N) \cdot c_{triple}$, where $n_{triples}$ counts on-grid incidences (sums landing on existing grid elements). The golden ratio maximizes capacity for $N \leq 5$ by maximizing the on-grid triple count.

The present sumset result uses the same algebraic identity but counts a different quantity. On-grid incidences ($\varphi^a + \varphi^{a+1} = \varphi^{a+2}$) identify a pair sum with a grid element but do not create sumset collisions, because no other pair produces the same sum. Sumset collisions ($\varphi^a + \varphi^{a+3} = 2\varphi^{a+2}$) identify two different pairs producing the same off-grid value. Both phenomena derive from $\varphi^2 = \varphi + 1$, but they measure orthogonal aspects of the additive structure: capacity measures how many sums land on the grid; the sumset measures how many sums coincide with each other.

6. Conclusion

Morphic numbers are the unique geometric ratios that produce sumset collisions. The golden ratio achieves the smallest sumset—and therefore the smallest doubling constant—among all geometric progressions for $N \leq 7$, with the plastic ratio taking over at $N \geq 8$. The exact formula $|G_\varphi + G_\varphi| = N(N+1)/2 - (N-3)$ follows from a single derived identity, $1 + \varphi^3 = 2\varphi^2$, that is itself a one-step consequence of the defining equation $\varphi^2 = \varphi + 1$. The morphic hierarchy's appearance in both channel capacity and sumset growth suggests that spectral closure is a structural phenomenon with broader combinatorial implications than its origins in information theory.

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