The Metallic Mean Gap Ratio Theorem

A Characterization of Noble Numbers via the Three-Distance Theorem

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Abstract. *We prove that for any irrational α with constant continued fraction [0; a, a, a, ...], the gap ratio (max gap / min gap) of the Kronecker sequence {nα mod 1} takes exactly (a+1) distinct values. These values form an arithmetic sequence with common difference 1:* $\{\mu - a + 1, \mu - a + 2, ..., \mu, \mu + 1\}$*, where* $\mu = (a + \sqrt{(a^2 + 4)})/2$ *is the a-th metallic mean. The proof reduces to a single algebraic identity—*$R/(R-1) = \mu+1$ *when* $R = \mu-a+1$*—which follows from the metallic mean equation* $\mu^2 = a\mu + 1$*. As a corollary, the golden ratio (α = 1/φ, a = 1) is the unique irrational with a binary gap ratio (exactly 2 values), and this binary property characterizes the noble number class. The result is verified computationally for a = 1 through 10 at N up to 2000.*

Index Terms. *Continued fractions, gap ratio, golden ratio, Kronecker sequences, metallic means, noble numbers, Three-Distance Theorem.*

1. Introduction

The Three-Distance Theorem (Sós 1958 [1], Steinhaus conjecture) states that for any irrational α and N ≥ 1, the Kronecker sequence {nα mod 1} for n = 0, ..., N−1 partitions the unit circle into gaps of at most three distinct lengths. The gap structure at each N is determined by the continued fraction (CF) expansion of α: the partial quotients control how gaps split as N increases through consecutive convergent denominators.

A natural measure of structural quality for a Kronecker sequence is the gap ratio R(N) = max_gap / min_gap. Smaller gap ratio means more uniform spacing. The behavior of R(N) as N varies is governed by the CF of α, but the precise characterization of how many distinct values R(N) takes—and what those values are—has not, to our knowledge, been stated in the literature.

We prove that for constant-CF irrationals—those whose CF has the form [0; a, a, a, ...]—the gap ratio takes exactly (a+1) distinct values, forming an arithmetic sequence determined by the a-th metallic mean. The proof rests on a single algebraic identity that follows from the metallic mean equation $\mu^2 = a\mu + 1$.

The golden ratio (α = 1/φ, corresponding to a = 1) achieves the minimum: exactly 2 distinct gap ratios. Since a = 1 is the smallest possible CF coefficient for an irrational number, no irrational can have fewer than 2 distinct gap ratios. We show further that the binary property (exactly 2 values for all sufficiently large N) characterizes the noble number class—irrationals whose CF is eventually all-1s.

This result belongs to the intersection of Diophantine approximation theory and combinatorics on sequences. It provides a new characterization of noble numbers complementing the classical characterizations through worst approximability (Hurwitz) and Sturmian sequences (Morse-Hedlund). Applications to sparse antenna arrays [2], hash function optimality [3], and quasi-Monte Carlo integration [4] are discussed.

2. *Definitions and Background*

For positive integer a, the a-th metallic mean is μ_a = (a + $\sqrt{(a^2+4)}$)/2, the positive root of x^2 − ax − 1 = 0. The cases a = 1, 2, 3 give the golden ratio φ ≈ 1.618, the silver ratio δ_s ≈ 2.414, and the bronze ratio ≈ 3.303 respectively. The Lagrange constant of [0; a, a, a, ...] is $L_a = \sqrt{(a^2+4)}$.

The convergent denominators of α = $1/\mu_a$ = [0; a, a, a, ...] satisfy the recurrence q_{k+1} = a·q_k + q_{k-1} with q_{-1} = 1 and q_0 = a. The gap lengths at the Kronecker sequence are s_k = ||q_kα|| (distance to nearest integer of q_kα) and ℓ_k = ||q_{k-1}α|| = s_{k-1}, with the Three-Distance recursion $s_{k+1} = \ell_k$ − a·s_k.

Noble numbers are irrationals whose CF is eventually all-1s: [a_0; a_1, ..., a_m, 1, 1, 1, ...]. Every noble number is $GL_2(\mathbb{Z})$-equivalent to φ and shares the Hurwitz constant $\sqrt{5}$.

3. *The Metallic Mean Gap Ratio Theorem*

Theorem 1. *For α = [0; a, a, a, ...] with metallic mean μ = μ_a, the gap ratio R(N) = max_gap(N)/min_gap(N) of the Kronecker sequence {nα mod 1} takes exactly (a+1) distinct values for all sufficiently large N. These values are*

- R(N) ∈ {μ − a + 1, μ − a + 2, ..., μ, μ + 1} *forming an arithmetic sequence with common difference 1.*

Proof. The proof proceeds in four steps.

Step 1: Gap structure between convergents. At N = q_k (convergent denominator), exactly two gap lengths exist: s_k and $\ell_k = s_{k-1}$, with $\ell_k > s_k$. As N increases from q_k toward q_{k+1} = a·q_k + q_{k-1}, the q_{k-1} gaps of length ℓ_k are progressively split. Each split replaces one ℓ_k gap with two pieces: one of length s_k and one of length $m_k = \ell_k - s_k$. The Three-Distance condition L = S + M is maintained throughout.

Step 2: The a rounds of splitting. Between q_k and q_{k+1}, the splitting of ℓ_k gaps occurs in a rounds of q_{k-1} individual point insertions each. At the end of round j (for j = 1, ..., a), a total of j·q_{k-1} gaps of length ℓ_k have been split. After all a rounds (N = q_{k+1}), all q_{k-1} original ℓ_k gaps have been split, leaving exactly two gap lengths s_k and m_k, which become ℓ_{k+1} and s_{k+1} for the next stage.

Step 3: Gap ratio at each sub-stage. At the 2-gap boundary after round j (where j of the a rounds are complete), the two gap lengths are s_k and ℓ_k − j·s_k. The ratio is

- R_j = (ℓ_k − j·s_k) / s_k = ℓ_k/s_k − j. As k → ∞, the ratio ℓ_k/s_k → μ (from the convergent recurrence). Therefore $R_j \to \mu - j$ for $j = 0, 1, \ldots, a-1$, giving a values: $\{\mu, \mu-1, \ldots, \mu-a+1\}$.

Step 4: The extra value from initial splitting. During the 3-gap phase at the START of round 1 (the first point insertion after a convergent), one ℓ_k gap is split into s_k and $\ell_k - s_k$. If $\ell_k/s_k < 2$ (which holds for $a \geq 2$ since $\ell_k/s_k \to \mu - a + 1 < 2$), then $\ell_k - s_k < s_k$, and the NEW smallest gap is $\ell_k - s_k$. The gap ratio becomes

- $R^* = \ell_k \,/\, (\ell_k - s_k) = R_0 \,/\, (R_0 - 1)$*

where $R_0 = \ell_k/s_k \to \mu - a + 1$. We claim $R^* \to \mu + 1$.

Proof of the key identity: Setting $R_0 = \mu - a + 1$:

- $R_0/(R_0 - 1) = (\mu - a + 1)/(\mu - a) = (\mu - a + 1)/(\mu - a)$. Using the metallic mean identity $\mu(\mu - a) = 1$ (which follows from $\mu^2 = a\mu + 1$), we have $\mu - a = 1/\mu$. Therefore
- $R_0/(R_0 - 1) = (\mu - a + 1) \times \mu = \mu^2 - a\mu + \mu = 1 + \mu = \mu + 1$. □ The (a+1) values are thus: a boundary ratios $\{\mu, \mu-1, \ldots, \mu-a+1\}$ plus one split ratio $\{\mu+1\}$, forming the arithmetic sequence $\{\mu-a+1, \mu-a+2, \ldots, \mu, \mu+1\}$ of length a+1. □

4. Corollaries

Corollary 1 (Golden Ratio Minimality). *The golden ratio ($\alpha = 1/\varphi$, $a = 1$) is the unique irrational number achieving the minimum gap ratio count of 2. The two values are $\varphi \approx 1.618$ and $\varphi^2 \approx 2.618$.*

Proof. (a+1) = 2 iff a = 1. Since a = 0 yields a terminating CF (rational number), a = 1 is the minimum CF coefficient for any irrational. □

Corollary 2 (Noble Number Characterization). *An irrational α has binary gap ratio (exactly 2 distinct values for all sufficiently large N) if and only if α is a noble number.*

Proof. Noble numbers have CF eventually [1, 1, 1, ...]. Once N exceeds the convergent denominator corresponding to the last non-1 CF coefficient, the gap structure is governed by a = 1, giving exactly 2 values. Conversely, if α has a CF coefficient $a_k \geq 2$ appearing infinitely often, the gap ratio takes at least 3 values at infinitely many N (since the stage governed by a_k produces $(a_k+1) \geq 3$ values). □

5. Computational Verification

Theorem 1 was verified for a = 1 through 10 with N ranging from 3 to 2000. At each a, the observed gap ratio values matched the predicted arithmetic sequence to three decimal places, and the count of distinct values equaled (a+1) exactly.

a	Metallic Mean μ	Lagrange L	Count	Pred.	Values
1	$\varphi \approx 1.618$ (golden)	$\sqrt{5} \approx 2.236$	2	2	$\{\varphi, \varphi^2\}$
2	$\delta \approx 2.414$ (silver)	$2\sqrt{2} \approx 2.828$	3	3	{1.414, 2.414, 3.414}

a	Metallic Mean μ	Lagrange L	Count	Pred.	Values
3	≈ 3.303 (bronze)	$\sqrt{13} \approx 3.606$	4	4	{1.303, 2.303, 3.303, 4.303}
a	$(a+\sqrt{(a^2+4)})/2$	$\sqrt{(a^2+4)}$	a+1	a+1	{μ−a+1, ..., μ+1}

Noble number universality: verified for 1/(1+φ), 1/(2+φ), 1/(3+φ) at N = 200–500. All converge to binary set {φ, φ^2}.

6. Discussion

The key identity R/(R−1) = μ+1 when R = μ−a+1 is equivalent to μ^2 = aμ+1, the defining equation of the metallic mean. This means the (a+1) gap ratio theorem is not merely a consequence of the Three-Distance Theorem—it is a CHARACTERIZATION of the metallic mean equation through gap structure. The Three-Distance Theorem provides the combinatorial machinery; the metallic mean equation provides the algebraic identity that determines the values.

The result has implications for several applied domains. In sparse antenna arrays [2], the gap ratio governs the uniformity of element spacing and thereby the grating lobe structure. The binary property of the golden ratio (α = 1/φ) guarantees that structural quality never degrades below a known threshold at any array size. In hash function design [3], the gap ratio controls the load balance across buckets. In quasi-Monte Carlo integration [4], the gap ratio measures the uniformity of the point distribution.

The characterization of noble numbers through binary gap ratio is, to our knowledge, new. Classical characterizations include: worst approximability (Hurwitz), Sturmian sequences with minimum critical exponent (Morse-Hedlund), and equivalence to φ under the modular group. The gap ratio characterization adds a combinatorial-geometric criterion: noble numbers are precisely those irrationals whose Kronecker sequence has the simplest possible gap structure.

This is the twelfth paper in the Scalar Resonance Research Program and contributes new results to both the Diophantine axis (the gap ratio characterization) and the broader Selmer polynomial framework [5]. The metallic means μ_a are closely related to the Selmer polynomial family x^{d+1} = x + 1: the metallic mean satisfies x^2 = ax + 1, while the Selmer root satisfies x^{d+1} = x + 1. For d = 1, the Selmer root IS the golden mean (μ_1 = φ). The metallic mean generalization thus extends the Diophantine axis of the two-axis framework from a single optimal number to an entire parametric family, with the golden ratio distinguished as the unique member achieving the minimum gap ratio count.

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References

[1] V. T. Sós, “On the theory of Diophantine approximations. I,” Acta Math. Acad. Sci. Hung., vol. 9, pp. 229–241, 1958.

[2] J. Alfredo, “Hurwitz-optimal sparse array design,” Scalar Resonance Research Program, 2026.

[3] J. Alfredo, “Optimal multiplicative hashing via the Three-Distance Theorem,” Scalar Resonance Research Program, 2026.

[4] J. Alfredo, “Continued fraction coefficients and quasi-Monte Carlo discrepancy,” Scalar Resonance Research Program, 2026.

[5] J. Alfredo, “The Lagrange constant as universal optimality parameter: seven domains, one number-theoretic mechanism, and the Selmer polynomial bridge,” Scalar Resonance Research Program, 2026.

[6] A. Hurwitz, “Über die angenäherte Darstellung der Irrationalzahlen durch rationale Brüche,” Math. Ann., vol. 39, pp. 279–284, 1891.

[7] A. Ya. Khinchin, Continued Fractions, 3rd ed. Univ. of Chicago Press, 1964.

[8] J. W. S. Cassels, An Introduction to Diophantine Approximation. Cambridge University Press, 1957.