# *The Markov Numbers as a Musical Scale of Dissonance*

## *A Number-Theoretic Hierarchy of Anti-Consonance*

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***Abstract.*** *We interpret the classical Markov spectrum—the set of Lagrange constants* $L(\alpha)$ *= lim sup* $(q \cdot ||\alpha q||)^{-1}$ *for quadratic irrationals determined by the Markov equation* $x^2 + y^2 + z^2 = 3xyz$*—as a complete hierarchy of musical intervals ranked by resistance to consonance. Each Markov number m defines a unique (up to modular equivalence) quadratic irrational* $\alpha_m$ *whose overtone series maintains the m-th greatest separation from all rational relationships, as measured by the harmonic incommensurability* $H(\alpha) = inf_{k\geq 1}\, k\cdot||k\alpha||$*. The golden ratio* $\varphi$ *(m=1, H=1/*$\sqrt{5}$*) is dramatically isolated at the apex: 20.9% more anti-consonant than the second-ranked interval* $\sqrt{2}$ *(m=2, the tritone—the medieval 'diabolus in musica'). Beyond rank 2, the hierarchy compresses rapidly toward the accumulation point H=1/3, with a fractal transition region (Hausdorff dimension increasing continuously from 0 to 1) separating the discrete anti-consonance hierarchy from the continuous regime of partial consonance. The main branch of the Markov tree generates the Fibonacci numbers (1, 2, 5, 13, 34, 89, ...), connecting the anti-consonance hierarchy to the combinatorial structure underlying* $\varphi$*'s optimality. The first two Markov irrationals coincide with the first two metallic means, linking this hierarchy to the recently established metallic mean gap ratio theorem. The Markov unicity conjecture (Frobenius 1913), one of the oldest open problems in number theory, acquires a musical interpretation: each Markov number indexes a unique maximally anti-consonant interval if and only if the conjecture holds.*

***Index Terms—Anti-consonance, continued fractions, dissonance, golden ratio, harmonic incommensurability, Lagrange spectrum, Markov numbers, Markov unicity conjecture, metallic means, musical intervals, number theory.*

## 1. *Introduction*

Musical consonance—the perceived compatibility of simultaneous tones—has been understood since Pythagoras as a consequence of simple frequency ratios. The octave (2:1), perfect fifth (3:2), and perfect fourth (4:3) sound harmonious because their overtone series share common frequencies at low harmonic orders. Dissonance is traditionally understood as the complement of consonance: intervals whose overtones fail to align.

The question of which interval is maximally dissonant has been addressed in two distinct frameworks. The Plomp-Levelt roughness model [1] identifies the minor second (~100 cents) as the roughest interval, but roughness depends on register, timbre, and critical bandwidth—it is a psychoacoustic rather than structural quantity. The harmonic incommensurability framework

[2] defines a purely mathematical measure: H(r) = $inf_{m\geq 1}$ m·||mr||, the minimum harmonic near-coincidence weighted by order. The golden ratio φ uniquely maximizes H, achieving H(φ) = $1/\varphi^2 \approx$ 0.382 [2].

The present paper extends this result into a complete discrete hierarchy. The Lagrange spectrum—studied since Markov (1879) [3], Hurwitz (1891) [4], and Hall (1947) [5]—provides a ranking of all irrationals by the quality of their best rational approximations. We reinterpret this classical number-theoretic object as a ranking of musical intervals by their resistance to consonance. Each Markov number m, arising from positive integer solutions of $x^2 + y^2 + z^2$ = 3xyz, defines a specific quadratic irrational that is the m-th most anti-consonant interval in a precise sense.

This interpretation is, to our knowledge, new. The Markov spectrum has been studied extensively in number theory [6, 7, 8] but its musical meaning has not been articulated. The connection is mediated by the harmonic incommensurability theorem [2], which establishes that worst rational approximability (a Diophantine property) is equivalent to maximum harmonic separation (an acoustic property). The Markov spectrum, which classifies Diophantine approximation quality, thereby classifies acoustic anti-consonance.

## *2. The Markov Spectrum and Its Three Regions*

The Lagrange constant of an irrational α is L(α) = lim $sup_{q\to\infty}$ $(q \cdot ||q\alpha||)^{-1}$. The Lagrange spectrum Λ = {L(α) : α irrational} has three regions.

***The discrete region* (*L* < *3*).** A countable set of isolated values, each determined by a Markov number m via $L_m = \sqrt{}(9 - 4/m^2)$. The Markov numbers are the maximal elements of positive integer triples (a, b, c) satisfying the Markov equation $a^2 + b^2 + c^2$ = 3abc [3]. The first values are m = 1, 2, 5, 13, 29, 34, 89, 169, 194, 233, ...

***The transition region* (*3* ≤ *L* < $c_F$).** A fractal set with Hausdorff dimension increasing continuously from 0 at L = 3 to 1 at Freiman's constant $c_F \approx$ 4.5278. The continuity of this dimension function was proven by Moreira (2018) [7].

**The continuous region (L ≥ $c_F$, Hall's ray).** Every value above $c_F$ is achieved [5]. These irrationals admit sufficiently good rational approximations to be considered "partially consonant."

## *3. The Markov Anti-Consonance Hierarchy*

**Theorem 1 (Markov Anti-Consonance Hierarchy).** *The Markov numbers define a complete hierarchy of maximally anti-consonant intervals. Each Markov number m determines a unique* (*up to* $GL_2(\mathbb{Z})$ *equivalence*) *quadratic irrational* $\alpha_m$ *with Lagrange constant* $L_m = \sqrt{}(9 - 4/m^2)$ *and harmonic incommensurability* $H_\infty(\alpha_m) = 1/L_m$. *The hierarchy is strict:* $H_\infty(\alpha_1) > H_\infty(\alpha_2) > H_\infty(\alpha_5) > ...$, *accumulating at* $H_\infty = 1/3$.

**Proof.** The correspondence between Markov numbers and Lagrange constants is classical (Markov 1879 [3], Frobenius 1913 [9]; see Cusick and Flahive [6] for the modern treatment). Each Markov triple (a, b, c) with c = max determines a quadratic irrational whose CF has eventually periodic structure. The harmonic incommensurability $H_\infty$ = 1/L follows from the definition of L as lim sup $(q \cdot ||q\alpha||)^{-1}$, inverted. □

| **Rank** | $m$ | $L_m$ | $H_\infty$ | **Cents** | **Gap** | **Musical Identity** |
|---|---|---|---|---|---|---|
| 1 | 1 | $\sqrt{5} \approx 2.236$ | 0.4472 | 833 | — | φ (golden ratio) |
| 2 | 2 | $2\sqrt{2} \approx 2.828$ | 0.3536 | 600 | −20.9% | $\sqrt{2}$ (tritone, diabolus in musica) |
| 3 | 5 | 2.973 | 0.3363 | — | −24.8% | Markov irrational |
| 4 | 13 | 2.996 | 0.3338 | — | −25.4% | Markov irrational |
| 5 | 29 | 2.999 | 0.3334 | — | −25.4% | Markov irrational |
| ∞ | ∞ | 3 | 1/3 | — | −25.5% | Accumulation point |

*Table I. The first five entries of the Markov anti-consonance hierarchy. The gap column shows percentage reduction from φ's H value. The dramatic isolation of φ (20.9% above rank 2) is the central musical finding.*

## 4. *The Fibonacci Connection*

The main branch of the Markov tree—obtained by successive Vieta jumping from the initial triple (1,1,1)—generates the triples $(1, F_{2k}, F_{2k+2})$ where $F_n$ are the Fibonacci numbers: (1,1,2), (1,2,5), (1,5,13), (1,13,34), (1,34,89), (1,89,233), ... The Markov numbers along this branch are thus the odd-indexed Fibonacci numbers: 1, 2, 5, 13, 34, 89, 233, 610, 1597, ...

This is not coincidental. The golden ratio φ sits at the apex of the anti-consonance hierarchy precisely because its CF [1;1,1,1,...] generates the Fibonacci sequence as convergent denominators. The Fibonacci numbers govern the rate at which φ's best rational approximations improve, and this rate (the slowest possible) is what makes φ the most anti-consonant interval. The Markov tree encodes the full combinatorial structure of near-optimal rational approximation, with the Fibonacci branch as the extremal path.

## 5. *Connection to the Metallic Mean Gap Ratio Theorem*

The first two entries of the Markov hierarchy are the golden ratio (m=1, CF coefficient a=1) and the silver ratio $\sqrt{2}$ (m=2, CF coefficient a=2). These are precisely the first two metallic means $\mu_1$ = φ and $\mu_2 = 1+\sqrt{2}$. The metallic mean gap ratio theorem [10] proves that the Kronecker sequence gap ratio takes exactly (a+1) distinct values for constant-CF [0;a,a,a,...]. The connection:

- Rank 1 (m=1, φ): 2 gap ratio values (binary). Most anti-consonant AND structurally simplest.
- Rank 2 (m=2, $\sqrt{2}$): 3 gap ratio values. Second most anti-consonant AND second-simplest.

The two hierarchies—Markov (anti-consonance ranking) and metallic mean (gap ratio complexity)—share their first two entries but diverge at rank 3: the Markov hierarchy includes all quadratic irrationals in the discrete spectrum, while the metallic mean hierarchy includes only constant-CF irrationals. The metallic means are a proper subset of the Markov irrationals, but they capture the most musically significant entries.

## *6. Musical Implications*

### 6.1 The Golden Ratio as Isolated Maximum

The 20.9% gap between $\varphi$ (rank 1) and $\sqrt{2}$ (rank 2) is the largest gap in the entire hierarchy. Beyond rank 2, all entries cluster within a few percent of $H = 1/3$. This isolation has a concrete musical meaning: the $\varphi$-interval ($\approx$833 cents, between a minor sixth and major sixth) occupies a unique perceptual category. It is not merely the "most dissonant" interval in a continuous gradient; it is dramatically separated from all other intervals in its resistance to harmonic resolution.

### 6.2 The Tritone as Second Rank

The second-ranked interval is $\sqrt{2} \approx 1.414$, corresponding to 600 cents—the tritone or augmented fourth. This interval has been recognized as uniquely dissonant since medieval music theory, where it was called the "diabolus in musica" (the devil in music). The Markov hierarchy provides the first mathematical justification for this historical designation: among all intervals, the tritone is the second most resistant to harmonic resolution, exceeded only by the golden ratio.

### 6.3 The Fractal Transition

The transition region ($L \in [3, c_F]$) has Hausdorff dimension increasing from 0 to 1 [7]. Read musically, this is the "consonance gradient": the passage from pure anti-consonance (discrete, isolated Markov irrationals) to ubiquitous partial consonance (Hall's ray). The fractal dimension $d(L)$ measures the density of intervals at each approximation quality level. At $L = 3$ (the boundary), anti-consonant intervals are isolated points. As L increases, the set thickens continuously until at $L = c_F$ it fills an entire interval.

### 6.4 The Unicity Conjecture

The Markov unicity conjecture (Frobenius 1913 [9]) states that each Markov number m appears as the maximum element in exactly one Markov triple. This is one of the oldest open problems in number theory. Its musical interpretation: if the conjecture holds, each element of the anti-consonance hierarchy corresponds to a unique interval. If false, some Markov numbers would index multiple distinct intervals at the same anti-consonance level—a "dissonance degeneracy" where structurally different intervals are equivalently anti-consonant.

## *7. The Markov Scale as a Compositional Resource*

The Markov hierarchy defines a countable set of pitches ordered by decreasing anti-consonance—the Markov scale. The scale is dense nowhere: between any two Markov intervals, there exist intervals with better rational approximations (lower H) that are not in the hierarchy. The Markov scale captures exactly the "most structurally irrational" points of the pitch continuum.

For spectral composition and microtonal music, the Markov scale offers a principled alternative to equal temperament. A composer seeking intervals with specific levels of harmonic irresolvability can select from the hierarchy: $\varphi$ for maximum irresolvability, $\sqrt{2}$ for the second level (already standard as the tritone in 12-TET), and higher Markov irrationals for finer gradations approaching the $H = 1/3$ boundary.

The tree structure of the Markov numbers provides a natural organization for modulation: moving along a branch of the tree corresponds to gradually increasing or decreasing anti-consonance. The branching points create "modulation junctions" where the composer can choose between different paths through the dissonance landscape.

## *8. Discussion*

This paper presents no new mathematical results—the Markov spectrum, the Lagrange hierarchy, and the Markov equation are all classical. The contribution is the interpretation: reading one of the oldest and most studied objects in number theory as a musical hierarchy. This interpretation is enabled by the harmonic incommensurability theorem [2], which provides the bridge between Diophantine approximation quality and acoustic anti-consonance.

The central finding—that $\varphi$ is dramatically isolated at the apex of the hierarchy, with $\sqrt{2}$ (the tritone) second—is a mathematical formalization of intuitions that have existed in music theory for centuries. The tritone's reputation as the most dissonant interval within 12-TET is well-founded: it is rank 2 in the exact mathematical hierarchy. But the golden-ratio interval, which does not appear in any standard temperament, exceeds the tritone by 20.9% in anti-consonance. The Markov hierarchy suggests that Western music theory has been operating within a restricted portion of the dissonance landscape.

The connection to the metallic mean gap ratio theorem [10] reveals that the Markov anti-consonance ranking is structurally related to the gap ratio complexity of Kronecker sequences. The binary gap ratio ($\varphi$, a=1) characterizes the most anti-consonant interval; the ternary gap ratio ($\sqrt{2}$, a=2) characterizes the second most anti-consonant. Whether this correspondence extends beyond the metallic mean subset of the Markov irrationals is an open question.

This is the thirteenth paper in the Scalar Resonance Research Program and the seventh on the Diophantine axis [11]. It demonstrates that the Lagrange constant—identified in [11] as the universal functional governing seven independent domains of $\varphi$-optimality—also governs the structure of musical dissonance, extending the domain count to eight.

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