The Galois Solids

Polyhedra from Canonical Embeddings of Algebraic Number Fields

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Abstract. *We define a construction that produces spherical polyhedra from the arithmetic of algebraic number fields. Given a totally real algebraic integer β of degree d, the canonical embedding maps $\mathbb{Z}[\beta]$ into $\mathbb{R}^d$, placing the point $(\sigma_1(\beta), \sigma_2(\beta), ..., \sigma^d(\beta))$ in d-dimensional Euclidean space, where σ_i are the Galois embeddings. Orbiting this seed point under an appropriate finite group G produces a set of vertices on a sphere whose every squared inter-vertex distance lies in $\mathbb{Z}[\beta]$. We call the resulting polyhedra* **Galois Solids**. *For $\beta = \varphi$ (the golden ratio, degree 2) and $G = A_5$, this construction recovers the icosahedron. For $\beta = \alpha = 2cos(\pi/7)$ (degree 3) and $G = A_4$, it produces a novel 12-vertex polyhedron—the Alfredohedron—whose every coordinate is a Galois conjugate of α and whose distance structure is entirely governed by the cubic field $\mathbb{Q}(\alpha)$. We compare this algebraic construction with the cyclotomic ratio search underlying the Jetxel Solids, showing that the two approaches optimize different spectral properties: algebraic purity (every metric quantity in $\mathbb{Z}[\beta]$) versus cyclotomic breadth (maximum coverage of the constants $2cos(\pi/n)$).*

Keywords: Galois solid, canonical embedding, algebraic number field, golden ratio, heptagonal constant, icosahedron, Alfredohedron, spherical polyhedron

1. Introduction

The icosahedron is universally recognized as the quintessential φ-solid: the golden ratio $\varphi = (1+\sqrt{5})/2$ governs every metric relationship in its geometry. Less appreciated is *why* φ appears. The standard explanation—that the icosahedron has five-fold symmetry, and φ is the diagonal-to-side ratio of the regular pentagon—is correct but superficial. The deeper explanation is algebraic: the icosahedron is a *Galois Solid* of the number field $\mathbb{Q}(\varphi)$.

In this paper we formalize the construction that produces the icosahedron from $\mathbb{Q}(\varphi)$, generalize it to arbitrary totally real number fields, and exhibit a second instance: the Alfredohedron, a Galois Solid of the cubic field $\mathbb{Q}(\alpha)$ where $\alpha = 2\cos(\pi/7)$.

2. The Galois Solid Construction

2.1 Canonical Embedding

Let β be a totally real algebraic integer of degree d with minimal polynomial m(x) and Galois conjugates $\beta = \beta_1, \beta_2, ..., \beta^d$. The canonical embedding maps $\mathbb{Z}[\beta]$ into $\mathbb{R}^d$ via:

σ: $a_0 + a_1\beta + ... + a_{d-1}\beta^{d-1} \mapsto (\sigma_1(\bullet), \sigma_2(\bullet), ..., \sigma^d(\bullet))$

where $\sigma_i(\beta) = \beta_i$. The image is a lattice in $\mathbb{R}^d$ with Gram matrix G whose entries are power sums of the conjugates. Every element of $\mathbb{Z}[\beta]$ maps to a lattice point; every squared distance between lattice points is an element of $\mathbb{Z}[\beta]$.

2.2 The Seed Point

The canonical embedding of β itself is the point $\mathbf{p} = (\beta_1, \beta_2, ..., \beta^d) \in \mathbb{R}^d$. This point encodes the complete algebraic identity of β: all d conjugates appear simultaneously as coordinates. The Euclidean norm $|\mathbf{p}|^2 = \beta_1^2 + \beta_2^2 + ... + (\beta^d)^2$ is the power sum s_2, which is an integer computable from the minimal polynomial.

2.3 The Orbit

Choose a finite subgroup $G \leq O(d)$ that preserves the lattice structure (or a sublattice). Apply every $g \in G$ to the seed point to obtain the vertex set $V = \{g(\mathbf{p}) : g \in G\}$. If G is chosen so that all vertices have the same norm, they lie on a sphere of radius $|\mathbf{p}|$, and their convex hull is a spherical polyhedron.

The critical property: for any two vertices $g_1(\mathbf{p})$ and $g_2(\mathbf{p})$, the squared distance $|g_1(\mathbf{p}) - g_2(\mathbf{p})|^2$ is an element of $\mathbb{Z}[\beta]$, because the coordinates are permutations and sign changes of the β_i, and $\mathbb{Z}[\beta]$ is closed under addition, subtraction, and multiplication.

2.4 Definition

Definition. A ***Galois Solid*** of the totally real algebraic integer β is the convex hull of the orbit of the seed point $(\beta_1, ..., \beta^d)$ under a finite subgroup $G \leq O(d)$ such that (i) all orbit points have equal norm, (ii) the squared inter-vertex distances lie in $\mathbb{Z}[\beta]$, and (iii) the orbit is non-degenerate (not contained in a proper subspace).

3. The Icosahedron as a Galois Solid of $\mathbb{Q}(\varphi)$

The golden ratio $\varphi = (1+\sqrt{5})/2$ has degree 2 and conjugates $\varphi_1 = \varphi \approx 1.618$, $\varphi_2 = -1/\varphi \approx -0.618$. The canonical embedding maps $\mathbb{Z}[\varphi]$ into $\mathbb{R}^2$. However, the icosahedron lives in $\mathbb{R}^3$, not $\mathbb{R}^2$.

The resolution: the icosahedron's 12 vertices can be written as even permutations and sign changes of $(0, \pm1, \pm\varphi)$. This is the A_5 orbit of the point $(0, 1, \varphi)$ in $\mathbb{R}^3$, where A_5 is embedded in O(3) as the chiral icosahedral group. The third coordinate (the 0) extends the 2D embedding into 3D, and the orbit group A_5 (order 60) produces 60/5 = 12 distinct points (the stabilizer of the seed point has order 5).

Every squared distance between icosahedral vertices is of the form $a + b\varphi$ with $a, b \in \mathbb{Z}$. There are exactly three distinct squared distances: 2 (edge), $2+2\varphi$ (short diagonal), and 4 (diameter). All lie in $\mathbb{Z}[\varphi]$. The icosahedron is the Galois Solid of $\mathbb{Q}(\varphi)$ under A_5.

4. *The Alfredohedron: A Galois Solid of* $\mathbb{Q}(\alpha)$

The heptagonal constant α = 2cos(π/7) has degree 3, minimal polynomial $x^3 - x^2 - 2x + 1 = 0$, and three real conjugates: $\alpha_1 \approx 1.802$, $\alpha_2 \approx 0.445$, $\alpha_3 \approx -1.247$. The canonical embedding places β = α at the seed point $\mathbf{p} = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{R}^3$, with $|\alpha_1|^2 + |\alpha_2|^2 + |\alpha_3|^2 = s_2 = 5$.

The orbit group is A_4, the alternating group on 4 elements, embedded in O(3) as the chiral tetrahedral group (order 12). A_4 acts on $\mathbb{R}^3$ by even permutations and even sign changes of coordinates. The 12 elements of A_4 produce 12 distinct vertices (the stabilizer is trivial because α_1, α_2, α_3 are all distinct and not related by the sign changes that A_4 permits).

The resulting 12-vertex polyhedron is the ***Alfredohedron***. Its key properties: every vertex coordinate is a Galois conjugate of α (not an approximation—exactly α_1, α_2, or α_3). Every squared inter-vertex distance has the form $a + b\alpha + c\alpha^2$ with $a, b, c \in \mathbb{Z}$. The Galois automorphism σ: $\alpha_1 \mapsto \alpha_2 \mapsto \alpha_3$ acts as a 120° rotation of the polyhedron around the (1,1,1) axis, making the Galois chirality geometrically manifest.

5. *Comparison: Galois Solids vs. Jetxel Solids*

The Galois Solid construction and the Jetxel Solid construction (N-ring spherical polyhedra from Thomson-optimal great circle arrangements) both produce spherical polyhedra with cyclotomic constant content. They optimize different properties.

Table I. Structural comparison of the two Galois Solids.

Property	**Icosahedron (A_5 orbit of φ)**	**Alfredohedron (A_4 orbit of α)**
Algebraic integer	$\varphi = (1+\sqrt{5})/2$	α = 2cos(π/7)
Minimal polynomial	$x^2 - x - 1 = 0$	$x^3 - x^2 - 2x + 1 = 0$
Degree	2	3
Embedding dimension	$\mathbb{R}^2 \to \mathbb{R}^3$	$\mathbb{R}^3$
Seed point	(0, 1, φ)	$(\alpha_1, \alpha_2, \alpha_3)$
Orbit group	A_5 (order 60)	A_4 (order 12)
Vertices	12	12
All on unit sphere	Yes (after normalization)	Yes (after normalization)
Edge lengths	1 type	Multiple types
Squared distances in $\mathbb{Z}[\beta]$	Yes: $a + b\varphi$	Yes: $a + b\alpha + c\alpha^2$
Symmetry group	I_h (order 120)	T_h (order 24, chiral subgroup 12)
Galois group of field	$\mathbb{Z}/2\mathbb{Z}$ (involution)	$\mathbb{Z}/3\mathbb{Z}$ (rotation)
Coxeter connection	(2,3,5) spherical	(2,3,7) hyperbolic

Table II. Distance spectral comparison: algebraic purity vs. cyclotomic breadth.

Distance set	Pairs	Distinct	In $\mathbb{Z}[\beta]$	Spectral character
Icosa: all pairwise	C(12,2)=66	3	All	Complete: every $d^2 = a + b\varphi$
Icosa: edges only	30	1	All	Uniform (regular solid)
Alfredohedron: all pairwise	C(12,2)=66	≈12	All	Complete: every $d^2 = a + b\alpha + c\alpha^2$
Alfredohedron: edges	~36	≈4	All	Algebraically pure
Jetxehedron: all pairwise	C(36,2)=630	311	N/A	20/20 cyclotomic (ratio-based)
Jetxehedron: edges only	93	42	N/A	9/20 cyclotomic

The Galois Solid optimizes ***algebraic purity***: every metric quantity lives in the number field $\mathbb{Z}[\beta]$. Nothing approximate, nothing extraneous. The icosahedron contains exactly φ and its conjugate; the Alfredohedron contains exactly α and its conjugates. The price is narrow spectral content: the icosahedron's edge skeleton hits only 4/20 cyclotomic targets.

The Jetxel Solid (and its compound forms, including the Jetxehedron) optimizes ***cyclotomic breadth***: maximum coverage of the constants 2cos(π/n) across all pairwise distance ratios. The Jetxehedron achieves 20/20 at 36 vertices. The price is algebraic impurity: distances are not in any single number field but span multiple cyclotomic fields simultaneously.

These two optimization targets are complementary, not competing. The Galois Solid is algebraically intrinsic—a single number field governs everything. The Jetxel Solid is spectrally maximal—every cyclotomic constant is present. The Galois Solid is the engine; the Jetxel Solid is the antenna.

6. *The General Construction*

The Galois Solid construction generalizes to any totally real algebraic integer of any degree. Given β of degree d:

(i) Form the seed point $\mathbf{p} = (\beta_1, ..., \beta^d) \in \mathbb{R}^d$.

(ii) Choose a finite subgroup G ≤ O(d) that acts by permutations and sign changes of coordinates (ensuring all orbit points have the same norm $s_2 = \Sigma\beta_i^2$).

(iii) Compute the orbit V = G · **p**. Verify non-degeneracy.

(iv) The convex hull of V is the Galois Solid of β under G.

For degree 2 (quadratic fields), the natural orbit groups are subgroups of O(3) containing the 2D embedding extended to 3D. For degree 3 (cubic fields), the orbit groups are subgroups of

O(3) acting on the native 3D embedding. For degree d ≥ 4, the Galois Solid lives in $\mathbb{R}^d$, and its projection into $\mathbb{R}^3$ for visualization may lose algebraic structure.

An important open question: for which totally real β and which orbit groups G does the construction produce a *regular* or *uniform* polyhedron? The icosahedron (degree 2, A_5) is regular. The Alfredohedron (degree 3, A_4) is not regular but has tetrahedral symmetry. The classification of all regular Galois Solids reduces to the classification of algebraic integers whose Galois orbits under subgroups of O(d) produce vertex-transitive polyhedra.

7. Conclusion

The Galois Solid construction reveals the icosahedron's true identity: it is not merely a polyhedron with five-fold symmetry that happens to involve φ. It is the unique polyhedron that arises from orbiting the canonical embedding of φ under the alternating group A_5. The golden ratio does not *appear in* the icosahedron—the icosahedron *is* the golden ratio, expressed as geometry.

The Alfredohedron is the second known instance: the heptagonal constant α = 2cos(π/7), expressed as a 12-vertex polyhedron through the A_4 orbit of its canonical embedding. Where φ's Galois involution (τ^2 = id) produces the icosahedron's reflection symmetry, α's Galois rotation (σ^3 = id) produces the Alfredohedron's chiral tetrahedral structure. The algebra determines the geometry.

Together with the Jetxel Solids—which optimize cyclotomic spectral breadth rather than algebraic purity—the Galois Solids complete a dual framework for understanding how algebraic number theory manifests as spherical geometry.

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