

The Galois Solids

Polyhedra from Canonical Embeddings of Algebraic Number Fields

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Abstract. We define a construction that turns the arithmetic of a number field into a spherical polyhedron. The canonical embedding of a totally real algebraic integer β of degree d is the point $(\sigma_1(\beta), \dots, \sigma_d(\beta))$ in $\mathbb{R}^d$, where the σ_i are the real embeddings. We orbit this point under a finite group of coordinate permutations and sign changes. When every conjugate of β lies in $\mathbb{Z}[\beta]$, as it does for φ and for every $2 \cos(\pi/n)$, so does each squared distance between two orbit points. We call the convex hull a **Galois Solid**. For $\alpha = 2 \cos(\pi/7)$ and the rotation group of the tetrahedron, the result is a new 12-vertex polyhedron, the Alfredohedron, whose coordinates are the conjugates of α themselves. It is chiral. Its 30 edges come in three lengths, and its 66 pairs of vertices fall into seven distance classes, all of them in $\mathbb{Z}[\alpha]$. The icosahedron has the same arithmetic property for $\mathbb{Z}[\varphi]$ and is our guiding example, although its usual seed $(0, 1, \varphi)$ is not a canonical embedding. We compare the construction with the Jetxel Solids, which aim at the whole family of constants $2 \cos(\pi/n)$ at once.

Keywords: Galois Solid, canonical embedding, algebraic number field, golden ratio, heptagonal constant, icosahedron, Alfredohedron, spherical polyhedron

Second edition. The first edition remains available, and the corrections are listed at the end.

1. Introduction

The golden ratio $\varphi = (1 + \sqrt{5})/2$ runs through the icosahedron. Place the icosahedron's twelve vertices at the cyclic permutations of $(0, \pm 1, \pm \varphi)$, and every squared distance between two of them lies in $\mathbb{Z}[\varphi]$. The usual explanation is five-fold symmetry, with φ as the ratio of diagonal to side in the regular pentagon.

This paper looks at the same fact from the arithmetic side: we ask which polyhedra have all their squared distances in one ring $\mathbb{Z}[\beta]$, and we build them directly from the Galois conjugates of β . Section 2 sets up the construction and the condition it needs. The icosahedron has the property without quite fitting the construction (Section 3); the Alfredohedron, built from the heptagonal constant $\alpha = 2 \cos(\pi/7)$, fits it exactly (Section 4).

2. The Galois Solid Construction

2.1 Canonical Embedding

Let β be a totally real algebraic integer of degree d , with minimal polynomial $m(x)$ and conjugates $\beta = \beta_1, \beta_2, \dots, \beta_d$. The canonical embedding sends $\mathbb{Z}[\beta]$ into $\mathbb{R}^d$:

$$\sigma: a_0 + a_1\beta + \dots + a_{d-1}\beta^{d-1} \mapsto (\sigma_1(\cdot), \sigma_2(\cdot), \dots, \sigma_d(\cdot)), \quad \sigma_i(\beta) = \beta_i.$$

The image is a lattice whose Gram matrix has the power sums of the conjugates as its entries. For $x, y \in \mathbb{Z}[\beta]$, the squared distance between $\sigma(x)$ and $\sigma(y)$ is the trace of $(x - y)^2$, which is an integer.

2.2 The Seed Point

The canonical embedding of β itself is the seed $\mathbf{p} = (\beta_1, \beta_2, \dots, \beta_d)$. All d conjugates appear together as its coordinates. Its squared norm is the power sum $s_2 = \beta_1^2 + \dots + \beta_d^2$, an integer that can be read off the minimal polynomial.

2.3 The Orbit

Choose a finite group $G \leq O(d)$ and apply each of its elements to the seed. The orbit $V = \{g(\mathbf{p}) : g \in G\}$ lies on the sphere of radius $|\mathbf{p}|$, because G is orthogonal, and its convex hull is a spherical polyhedron.

Suppose G acts by permuting coordinates and changing their signs. The squared distance between two orbit points is then a polynomial with integer coefficients in $\beta_1, \dots, \beta_d$, so it lies in $\mathbb{Z}[\beta]$ as soon as every conjugate β_i does. For φ the conjugate is $1 - \varphi$, and for $\alpha = 2\cos(\pi/7)$ the conjugates are $\alpha^2 - \alpha - 1$ and $2 - \alpha^2$. The same goes for every $2\cos(\pi/n)$, because each $2\cos(k\pi/n)$ is an integer polynomial in $2\cos(\pi/n)$. Without this condition the property can fail. Take β to be the largest root of $x^3 - 4x - 1$, whose field is not Galois over $\mathbb{Q}$: in the orbit of Section 4's group, four of the seven squared distances lie outside $\mathbb{Z}[\beta]$.

2.4 Definition

Definition. A *Galois Solid* of the totally real algebraic integer β is the convex hull of the orbit of the seed $(\beta_1, \dots, \beta_d)$ under a finite subgroup $G \leq O(d)$ such that (i) all orbit points have equal norm, (ii) the squared distances between vertices lie in $\mathbb{Z}[\beta]$, and (iii) the orbit is not contained in a proper subspace.

Condition (i) holds for every orthogonal G . Condition (ii) is the one that carries the arithmetic, and by Section 2.3 it holds for coordinate permutations and sign changes whenever the conjugates of β lie in $\mathbb{Z}[\beta]$.

3. The Icosahedron and $\mathbb{Q}(\varphi)$

The golden ratio has degree 2, with conjugates $\varphi_1 = \varphi \approx 1.618$ and $\varphi_2 = -1/\varphi \approx -0.618$. Its canonical embedding lives in the plane. The icosahedron lives in $\mathbb{R}^3$: its twelve vertices, the

cyclic permutations of $(0, \pm 1, \pm \varphi)$, form the orbit of $(0, 1, \varphi)$ under the rotation group A_5 of order 60. The stabilizer of $(0, 1, \varphi)$ has order 5, which leaves $60/5 = 12$ points.

Every squared distance between these vertices has the form $a + b\varphi$ with $a, b \in \mathbb{Z}$, and there are exactly three of them: 4 between neighbors (30 pairs), $4 + 4\varphi$ between vertices two steps apart (30 pairs), and $8 + 4\varphi$ between opposite vertices (6 pairs). They stand in the ratios $1 : \varphi^2 : \varphi^2 + 1$.

The seed $(0, 1, \varphi)$ is not the canonical embedding of φ , which would be the plane point $(\varphi, -1/\varphi)$, so strictly the icosahedron is not a Galois Solid under the definition above. It has the defining property, with all its squared distances in $\mathbb{Z}[\varphi]$, and it is the example the construction was built to extend.

4. The Alfredohedron: A Galois Solid of $\mathbb{Q}(\alpha)$

The heptagonal constant $\alpha = 2 \cos(\pi/7)$ has degree 3 and minimal polynomial $x^3 - x^2 - 2x + 1$. Its three conjugates are real: $\alpha_1 \approx 1.802$, $\alpha_2 \approx 0.445$ and $\alpha_3 \approx -1.247$. The field $\mathbb{Q}(\alpha)$ is Galois over $\mathbb{Q}$, with a cyclic Galois group of order 3, and both other conjugates lie in $\mathbb{Z}[\alpha]$: $\alpha_2 = \alpha^2 - \alpha - 1$ and $\alpha_3 = 2 - \alpha^2$. The seed is $\mathbf{p} = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{R}^3$, with $|\mathbf{p}|^2 = s_2 = 5$.

The orbit group is A_4 , acting as the rotation group T of the tetrahedron, of order 12: its elements permute the three coordinates cyclically and change the signs of an even number of them. The stabilizer of $\mathbf{p}$ is trivial because the conjugates are distinct and nonzero, so the orbit has 12 points.

The convex hull of these points is the *Alfredohedron*. Each vertex coordinate is exactly $\pm\alpha_1, \pm\alpha_2$ or $\pm\alpha_3$, and the surface has 20 triangular faces and 30 edges, the same counts as the icosahedron, but the edges come in three lengths, with squared lengths $10 - 4\alpha \approx 2.792$, $20 - 4\alpha^2 \approx 7.012$ and $14 + 4\alpha - 4\alpha^2 \approx 8.220$. Over all 66 pairs of vertices there are seven squared distances, each of the form $a + b\alpha + c\alpha^2$ with integers a, b, c (Table II).

The solid is chiral. Its full symmetry group is T itself: it has no reflection symmetry and no center of inversion, so it cannot be superposed on its mirror image. Applied to the seed, the Galois automorphism $\alpha_1 \mapsto \alpha_2 \mapsto \alpha_3 \mapsto \alpha_1$ gives the same point as the rotation by 120° about the axis $(1, 1, 1)$, and that rotation is one of the solid's symmetries. Applied to every coordinate of every vertex, the automorphism still permutes the twelve vertices, but no single rotation of the solid does the same.

5. Comparison: Galois Solids and Jetxel Solids

The Jetxel Solids are spherical polyhedra built from Thomson-optimal arrangements of great circles; like Galois Solids they carry cyclotomic constants in their distances, but they are tuned for a different goal. Table I sets the icosahedron and the Alfredohedron side by side, and Table II adds the Jetxhedron, the Jetxel Solid with the smallest total miss.

A Galois Solid is exact: every metric quantity lies in one ring $\mathbb{Z}[\beta]$. It reaches few of the cyclotomic constants, though. Among the twenty that the Jetxel Solids aim for, $2 \cos(\pi/n)$ for $n = 3, \dots, 20$ together with φ and $1/\varphi$, the icosahedron's distance ratios supply only $\varphi, 1/\varphi$ and $2 \cos(\pi/10)$.

The Jetxhedron, a 42-vertex polyhedron with 84 edges and 44 faces, comes within a total of

Table I. Structural comparison of the icosahedron and the Alfredohedron.

Property	Icosahedron	Alfredohedron
Algebraic integer	$\varphi = (1 + \sqrt{5})/2$	$\alpha = 2 \cos(\pi/7)$
Minimal polynomial	$x^2 - x - 1$	$x^3 - x^2 - 2x + 1$
Degree	2	3
Seed	$(0, 1, \varphi)$, see Section 3	$(\alpha_1, \alpha_2, \alpha_3)$
Orbit group	A_5 (order 60)	A_4 (order 12)
Vertices, edges, faces	12, 30, 20	12, 30, 20
Edge lengths	one	three
Squared distances	$a + b\varphi$	$a + b\alpha + c\alpha^2$
Symmetry group	I_h (order 120)	T (order 12), chiral
Galois group of the field	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$
Coxeter connection	$(2, 3, 5)$, spherical	$(2, 3, 7)$, hyperbolic

Table II. Distance classes. The Jetxhedron figures are those of its source paper [2].

Solid	Vertices	Pairs	Distinct distances	Exact ring
Icosahedron	12	66	3	$\mathbb{Z}[\varphi]$
Alfredohedron	12	66	7	$\mathbb{Z}[\alpha]$
Jetxhedron	42	861	about 421	none

0.000192 of all twenty targets [2], but its distances do not lie in any single number field.

6. The General Construction

Given β of degree d :

- (i) form the seed $\mathbf{p} = (\beta_1, \dots, \beta_d) \in \mathbb{R}^d$;
- (ii) choose a finite group G of coordinate permutations and sign changes, so that every orbit point has norm $\sqrt{s_2}$;
- (iii) compute the orbit $V = G \cdot \mathbf{p}$ and check that it spans $\mathbb{R}^d$;
- (iv) take the convex hull of V .

The result is a Galois Solid when the conjugates of β lie in $\mathbb{Z}[\beta]$ (Section 2.3). That includes every $2 \cos(\pi/n)$, the constants the Jetxel Solids are built around. For $d = 2$ it is a polygon in the plane, which is why the icosahedron needs a seed of its own; for $d = 3$ it lives in $\mathbb{R}^3$ directly, and for $d \geq 4$ in $\mathbb{R}^d$, where a projection to three dimensions may lose part of the arithmetic.

An open question is which β and which G give a regular or uniform Galois Solid. Since the vertices form a single orbit, G carries every vertex to every other, and the question is only whether all edges have the same length and all faces are regular. The icosahedron, outside the strict definition, is regular. The Alfredohedron is not, since its edges have three lengths.

7. Conclusion

Orbiting the conjugates of an algebraic integer under a group of signed permutations gives polyhedra whose whole distance structure lies in one ring, provided the conjugates lie in that ring, as they do for the golden ratio and for every $2 \cos(\pi/n)$. The Alfredohedron is the instance built from $2 \cos(\pi/7)$: a chiral solid whose seven distance classes all lie in $\mathbb{Z}[\alpha]$. The Jetxel Solids pursue the other goal, near-coverage of every $2 \cos(\pi/n)$ at once.

Changes from the first edition

1. Section 2.3. The first edition said that every squared distance lies in $\mathbb{Z}[\beta]$ for any totally real β . This needs the conjugates of β to lie in $\mathbb{Z}[\beta]$; the paper now says so and gives a cubic for which it fails.
2. Section 3. The icosahedron's squared distances at the vertices $(0, \pm 1, \pm \varphi)$ are $4, 4 + 4\varphi$ and $8 + 4\varphi$; the first edition printed $2, 2 + 2\varphi$ and 4 , which no rescaling reconciles.
3. Section 3 and Section 7. The seed $(0, 1, \varphi)$ is not the canonical embedding of φ . The claims that the icosahedron is the unique polyhedron obtained from that embedding, and that the Galois involution of φ produces its reflections, are withdrawn.
4. Section 4. The Galois automorphism agrees with the 120° rotation about $(1, 1, 1)$ on the seed only. Across all vertices it permutes them without being a rotation.
5. Table I. The Alfredohedron's symmetry group is T , of order 12, with no reflections (the first edition gave T_h , of order 24).
6. Table II. The Alfredohedron has 7 distance classes and 30 edges of 3 lengths (the first edition gave about 12 classes, about 36 edges and about 4 lengths). The Jetxhedron has 42 vertices, 84 edges and about 421 distance classes, as in its source paper; the first edition's row for a 36-vertex solid had no source.
7. Section 6. The first edition reduced the classification of regular Galois Solids to vertex-transitive polyhedra. Every Galois Solid is vertex-transitive under its own group, so that reduction was empty; the question is restated.

References

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